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ABSTRACT

This paper concerns with the queueing model GI/M/m with single arrivals and batch service. The capacities of service $\{\eta_n\}$ are i.i.d. r.v's. Let $\{\tau_n\}$ be the arriving instants, we take $\{\tau'_n\} = \{\tau_{\sum_{i=1}^{n} \eta_i}\}$ to be the imbedded points and the number of batch served and waiting to be the states of systems. We have found : <i> the limiting distribution queue length (number of batch served and waiting) and that of batch waiting in the queue; <ii> for "first-come-first-served", the limiting distribution function of waiting time of batch, and that of a customer chosen randomly in batch; for "served in random order" and "last-come-first-served", the Laplace-Stieltjes transformation of the above distributions.

RESUME

On étudie dans cet article le système GI/M/m avec services par groupes, arrivées ponctuelles simples (processus de renouvellement). Les différentes variables aléatoires mises en jeu sont indépendantes.

On calcule la distribution stationnaire du nombre de groupes en attente et en service. On obtient également la transformée de Laplace-Stieltjes du temps d'attente (d'un groupe et d'un client arbitraire dans le groupe) pour les disciplines de service PAPS, DAPS et "Aléatoire".

Note : Cet article a été effectué en 1985 à l'occasion du séjour du Professeur Chen Yong Yi à l'INRIA dans le projet MEVAL sous la direction de Guy Fayolle.

INTRODUCTION

In this paper we shall consider the queueing model GI/M/m with single arrivals and batch service. For the queueing system with batch service, there are many different detailed disciplines (cf. [3]), one of which is: if the number of customers in the queue is less than the capacity of service, all of the present customers are served. For this case, the two situations are distinguished: <i> the customers arriving during the service are also served so that the capacity of service may be complemented (cf. [1], for GI/M/1); <2> the customers arriving later will be served in the following batches (cf. [1] for M/G/1; [2] for GI/M/S). Another discipline is: if and only if the number of customers in the queue is not less than the capacity of service, the customers are served. (cf. [4] for M/G/1).

In this paper we consider only <2>, and the capacities of service are independent random variables with the common distribution (i.i.d. r.v's). For system GI/M/S with fixed capacity of service (according to the discipline <1>), the complet results are obtained in [2].

DESCRIPTION OF SYSTEM

1. The customers arrive at the instants $\tau_0, \tau_1, \tau_2, \ldots, \tau_n, \ldots$ where the interarrival times $\tau_{n+1}-\tau_n$ $(n=0,1,\ldots,\tau_0 = 0)$ are identically distributed, independent, positive random variables with the distribution function

$$P(\tau_{n+1}-\tau_n \leq x) = F(x), \tag{2.1}$$

its expectation

$$\beta = \int_0^\infty x dF(x) < \infty,$$

and its Laplace-Stieltjes transformation is $\phi(s)$.

2. There are m servers and the order of service is irrelevant. The service times are identically distributed, independent random variables with distribution function

$$H(x) = \begin{cases} 1-e^{-\mu x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \tag{2.2}$$

3. We denote the capacity of service of the n-th batch service by η_n. $\{\eta_n\}_{n\geq 1}$ are identically distributed, independent, integral random variables with the generating function

$$Ez^{\eta_n} = \sum_{i=1}^{\infty} \phi_i z^i, \tag{2.3}$$

and its expectation

$$\sum_{i=1}^{\infty} i\phi_i = d < \infty.$$

All of the above random variables are mutually independent. If the number of customers in the queue at the moment that the idle server can start with the (n-1)-th batch service is not less than η_n, then η_n customers are immediately served. If this number is less than η_n, then waiting until the number of customers in the queue attains to η_n, the customers are served.

According to the traditional method of imbedded Markov chain the imbedded points are τ_n, and the states are $\xi_n = \xi(\tau_n - 0)$, n=0,1,2,.... In this paper we take a subset of $\{\tau_n\}$ to be the imbedded points, and the states of system are the number of batches of customers (served and waiting) in the system. (also cf. [3]). By this way, we can successfully consider this system. Concretely speaking, we choose a subset of $\{\tau_n\}$,

$$\tau_0, \tau_{\eta_1}, \tau_{\eta_1+\eta_2}, \ldots, \tau_{\sum_{i=1}^{k}\eta_i}, \ldots, \tag{2.4}$$

where $\tau_{\sum_{i=1}^{k} \eta_i}$, $k=1,2,\ldots$, is the arriving instant of the last customer of the k-th batch of customers. We denote $\tau_{\sum_{i=1}^{k} \eta_i}$ by τ'_k, $k=1,2,\ldots$, $\tau'_0 \overset{def}{=} 0$.

Evidently, $\tau'_{n+1}-\tau'_n$ is the sum of η_{n+1} independent random variables whose common distribution function is F(x). Denoting the distribution function of $\tau'_{n+1}-\tau'_n$ by $\tilde{F}(x)$, we obtain

$$\tilde{F}(x) = \sum_{i=1}^{\infty} \phi_i F^{(i)}(x), \tag{2.5}$$

the Laplace-Stieltjes transform of $\tilde{F}(x)$

$$\tilde{\Phi}(x) = \sum_{i=1}^{\infty} \phi_i [\Phi(x)]^i \tag{2.6}$$

and its expectation $d\beta$, where $\Phi(x)$ is the Laplace-Stieltjes transform of F(x), $F^{(i)}(x)$ is the i-fold convolution of F(x).

Denote the number of batches of customers (waiting or being served) at the instant τ'_n+0 by ξ_n, i.e.

$$\xi_n = \xi(\tau'_n+0), \qquad n = 0,1,2,\ldots \tag{2.7}$$

$\{\xi_n\}_{n\geq 0}$ take the positive integers, only ξ_0 may be 0. When $\xi_0 = 0$, then $\&_1 = 1$ with probability 1, and after t=1, this chain take not ever 0.

Hence below we shall consider {1,2,...} as the state space. If $\xi_0 = 0$, we may transform this problem into that of the original state $\xi_1 = 1$. We indicate that $\xi(t)$ is generally insensible, since $\forall\ t \geq 0$, the number of customers in the system takes uncertainly form of $\{\sum_{i=1}^{k} \eta_i\}$.

According to an analogous way to [2], we can easily prove $\{\xi_n\}_{n\geq 0}$ is an irreducible, aperiodic imbedded Markov chain. By means of the above method, this problem considered is transformed into an ordinary GI/M/m queueing system with the distribution function of interarrival times $\tilde{F}(x)$ and $\xi_n = \xi(\tau'_n + 0)$. From a simple modification of the well-known results of GI/M/m (cf. [4]), we obtain the following theorems :

<u>Theorem 1</u> If $md\beta\mu > 1$, then the limiting distribution

$$\lim_{n\to\infty} P\{\xi_n = k\} = P_k, \quad k = 1, 2, \ldots$$

exists and it is independent of the original distribution $\{P(\xi_0 = k)\}_{k\geq 0}$, where

$$P_k = \begin{cases} \sum_{r=k-1}^{m-1} (-1)^{r-(k-1)} \binom{v}{k-1} U_r, & k = 1, 2, \ldots, m, \\ A\, \omega^{k-(m+1)}, & k = m+1, m+2, \ldots, \end{cases}$$

$$U_r = AC_r \sum_{j=r+1}^{m} \frac{\binom{m}{j}}{C_j(1-\phi_j)} \left(\frac{m(1-\phi_j)-j}{m(1-\omega)-j}\right),$$

$$A = \left[\frac{1}{1-\omega} + \sum_{j=1}^{m} \frac{\binom{m}{j}}{C_j(1-\phi_j)} \left(\frac{m(1-\phi_j)-j}{m(1-\omega)-j}\right)\right]^{-1} \tag{2.8}$$

$$C_j = \prod_{v=1}^{j} \left(\frac{\phi_v}{1-\phi_v}\right),$$

$$\phi_v = \tilde{\Phi}(v\mu),$$

ω is the only root of the equation

$$z = \tilde{\Phi}(\mu m(1-z))$$

in the unit circle, and ω is the smallest absolute value, positive real root; $\omega < 1$.

If $md\beta\mu \leq 1$, then $\omega = 1$, and

$$\lim_{n\to\infty} P\{\xi_n = k\} = 0, \quad k = 1,2,\ldots. \tag{2.9}$$

The original state ξ_0 is considered as the number of batches arriving in the system before $t=0$, and we assume that the capacities of batches of service $\{\eta_i\}_{-\infty<i\leq 0}$ are independent of batches of service of customers arriving before $t=0$.

Proof For the ordinary GI/M/m queueing system, putting $\xi_n^* = \xi(\tau_n - 0)$ and $\xi_n = \xi(\tau_n + 0)$ respectively, it is easily seen that

$$(\xi_n = k) = (\xi_n^* = k-1), \quad k=1,2,\ldots \,. \tag{2.10}$$

From the results of [4] and (2.10), we can immediately prove this theorem.

Corollary In the stationary state, the average queue length (number of batches waiting and served) equals

$$1 + U_1 + A\,\frac{\omega + m(1-\omega)}{(1-\omega)^2}, \tag{2.11}$$

and the distribution function of the actual queue length (number of batches waiting) is

$$P\{\hat{q}=j\} = \begin{cases} 1 - \dfrac{A\omega}{1-\omega}, & j = 1 \\ A\omega^{j-1}, & j = 2,3,\ldots, \end{cases} \tag{2.12}$$

and its average equals

$$1 + \frac{A\omega}{(1-\omega)^2}. \tag{2.13}$$

Proof Noting that we have (2.10) for ξ_n^* and ξ_n, it is clear that the queue length of ξ_n equals that of ξ_n^* plus 1. Again utilizing the results of [4], we can obtain (2.11). From (2.10) and [4], we obtain easily (2.12) and (2.13).

Theorem 2 For the case "first-come-first-served", when $md\beta\mu > 1$, the limiting distribution function of waiting times of batch, i.e. that of waiting times of the last customer of batch is

$$W_B(x) = 1 - \frac{Ae^{-m\mu(1-\omega)x}}{1-\omega}, \quad x \geq 0, \tag{2.14}$$

its expectation is

$$\frac{A}{(1-\omega)^2 m\mu}. \tag{2.15}$$

The limiting distribution function of waiting time of a customer chosen in random order in batch is

$$W_R(x) = \sum_{j=1}^{\infty} \sum_{i=1}^{j} \frac{\Phi_j}{j} W_B(x) * F^{(i-1)}(x), \tag{2.16}$$

and its expectation is

$$\int_0^{\infty} x dW_R(x) = \frac{A}{(1-\omega)^2 m\mu} + \frac{\beta(d-1)}{2} \tag{2.17}$$

Proof Noting that whether $\xi_n = \xi(\tau_n - 0)$ or $\xi_n = \xi(\tau_n + 0)$, for the ordinary GI/M/m queueing system, the waiting time of a customer arriving in the system is same, we obtain easily from [4] the formulas (2.14) and (2.15).

Below we shall prove (2.16) and (2.17). Noting that if the capacity of service of a batch is j with the probability ϕ_j, a customer chosen in random order in this batch is the first arrival with probability $\frac{1}{j}$, or the second arrival with probability $\frac{1}{j}$,..., or the j-th arrival with probability $\frac{1}{j}$, we prove easily (2.16).

From (2.16) and (2.15) we obtain

$$\int_0^\infty x dW_R(x) = \sum_{j=1}^\infty \sum_{i=1}^j \frac{\phi_j}{j} \{\frac{A}{(1-\omega)^2 m\mu} + (i-1)\beta\} = \frac{A}{(1-\omega)^2 m\mu} + \frac{\beta}{2}(d-1).$$

It seems that the average waiting time of a customer chosen in random order in batch is also an interesting effective measure of system.

<u>Theorem 3</u> For the case "served in random order", when $md\beta\mu > 1$, the Laplace-Stieltjes transform of the limiting distribution of waiting time of batch is

$$\Omega_B(s) = \int_0^\infty e^{-sx} dP\{W \le x\} = 1 - \frac{A}{1-\omega} + A\sigma(s,\omega), \tag{2.18}$$

where $\sigma(s,z)$ satisfies the following linear differential equation

$$\{z-\tilde{\Phi}(s+m\mu(1-z))\} \frac{\partial\sigma(s,z)}{\partial z} + \sigma(s,z) = \frac{m\mu\{1-\tilde{\Phi}(s+m\mu(1-z))\}}{(1-z)(s+m\mu(1-z))}$$

and

$$\sigma(s,\gamma(s)) = \frac{m\mu}{s+m\mu(1-\gamma(s))},$$

where $\gamma(s)$ is the only solution of the following equation

$$z = \tilde{\Phi}(s+m\mu(1-z)), \quad R(s) \ge 0$$

in the unit circle; and $\omega = \gamma(0)$, A is defined by (2.8). The Laplace-Stieltjes transform of limiting distribution of waiting time of a customer chosen randomly in batch is

$$\tilde{\Omega}_R(s) = \Omega_B(s) \sum_{j=1}^\infty \frac{\phi_j}{j} [\phi(s)]^{j-1}, \tag{2.19}$$

its average waiting time is defined by (2.17).

Proof Noting that for the ordinary GI/M/m queueing system, whether $\xi_n=\xi(\tau_n-0)$ or $\xi_n=\xi(\tau_n+0)$, the waiting time of a customer arriving in the system is same, it follows from [5] that (2.18) holds. By the similar way as the proof of (2.16), we prove easily (2.19).

Since for the ordinary GI/M/m queueing system, whether "first-come-first-served" or "served in random order", the average waiting time is same, the average waiting time of a customer chosen randomly in batch is defined by (2.17).

Theorem 4 For the case "last-come-first-served", the Laplace-Stieltjes transform of limiting distribution of waiting time of batches is

$$\Omega(s) = \int_0^\infty e^{-sx} dP(W\le x) = 1 - \frac{A}{1-\omega} + \frac{A}{1-\omega}\,\frac{m\mu(1-\gamma(s))}{s+m\mu(1-\gamma(s))}, \qquad (2.20)$$

wher $A,\omega,\gamma(s)$ are the same as these of theorem 3; its expectation is the same as (2.15).

The Laplace-Stieltjes transform of waiting time of a customer chosen randomly in batch is

$$\tilde{\Omega}(s) = \Omega(s) \sum_{j=1}^{\infty} \frac{\Phi_j}{j} [\Phi(s)]^{j-1}, \qquad (2.21)$$

where $\Omega(s)$ is defined by (2.20), and its expectation is the same as (2.17).

Proof By the similar way as the proof of theorem 3, one easily proves theorem 4.

Note Above we denote the capacities of service of the n-th batch service by η_n. In fact, this is correct only if "first-come-first-served". For "served in random order" and "last-come-first-served" etc. it is uncertain that η_n is the capacity of the n-th batch service and that $\tau'_n = \sum_{i=1}^{n} \eta_n$. In the case of "served in random order "(or "last-come-first-served"), one divides the customers into groups in arriving order by the following manner : taking the first n_1 arriving customers to be a group; the (n_1+1)-th to (n_1+n_2)-th customer to be an other group, etc. The customers of same group are served in same time. When one starts a service, a group chosen randomly (or the last arriving group) in waiting groups is served.

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